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Coinage, debasements, and Gresham's laws★

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Summary. This paper formulates a model of commodity money that circulates by tale, and applies it to a variety of situations, some of which seem to confirm, and others to contradict, 'Gresham's Law'. We analyze how debasements could prompt decisions of citizens voluntarily to participate in recoinages that subjected them to seigniorage taxes.

JEL Classification Numbers: E60.

1 Introduction

For hundreds of years, supplies of coins in Europe emerged, via a curious mechanism, from voluntary decisions of owners of old coins and bullion to exchange them at mints for new coins. Mints were sometimes private enterprises, licensed to produce on demand a list of coins whose design and fineness was specified by the sovereign. Citizens were free to take metal to the mint[1] to purchase newly minted coins, often bearing less rare metal than was surrendered for them. Private mints covered their costs, which included a per-coin seigniorage tax. Under this mechanism, rates of coinage, and therefore also seigniorage revenues, fluctuated over time, with low rates of minting often being accompanied by complaints about shortages of small denomination coins, as well as physical depreciation of the currency.

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[1] Even when mints were government operated, the government typically had no power to compel agents to engage in coinage.

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A *debasement* was a respecification of the list of coins which could be produced, with the rare metal content (the fineness) of coins of each nominal value being reduced. Acts of debasement generated revenues for the sovereign only when they prompted voluntary increases in demand for newly minted coins. As revenue raising devices, debasements often 'worked,' being followed by spurts in minting. It seems a puzzle that debasements should have provoked *voluntary* decisions of owners of metal to line up to pay seigniorage taxes by surrendering more substantial coins for less; it certainly is a puzzle if we use a model of the demand for money in which coins circulated by *weight*.[2]

In this paper, we use a model in which coins circulate by face value or *tale* to explain this behavior. Modelling circulation by tale elucidates this 'debasement puzzle' (a phrase of Rolnick, Weber, and Velde) as well as various doctrines in monetary economics involving Gresham's Law (or, as we shall see, *Laws*) and the quantity theory of money. A 'small country' version of our model has coins circulating by tale domestically and by weight internationally, an interpretation that agrees with the descriptions of Smith [27], Fisher [12], and Jevons [17].[3]

A system of commodity money in which coins are valued by tale occupies intermediate ground between a fiat money system and one where coins are valued according to their weight. The array of devices that support the value of a fiat currency are also available to push the value of a commodity money above that of the metals that compose it. This observation raises the possibility that under-weight coins could at times bear values high enough to coexist with full-weight coins. The issue of whether and when under-weight coins drive out full weight coins propels us into the thicket of doctrines occupied by Gresham's Law.

We adapt Lucas's [18] cash-in-advance model to our task by altering the conditions of money creation to capture key features of the minting process.[4] Lucas studied a situation in which cash was created by government fiat and supplied via a particular class of government fiscal policies. We replace Lucas's model of supply with a mechanism that makes the supply of money endogenous, including: (a) a technology for minting and melting coins,[5] (b) a

[2] Rolnick, Velde, and Weber [23] summarize a set of observations from medieval monetary regimes which are "... all consistent with the following stylized fact: large debasements are accompanied by unusually large minting volumes that yield unusually large revenues for the sovereign." (p. 1) They frame as a puzzle the fact that large debasements *attracted* voluntary deliveries of metal to the mints, while small ones often did not.

[3] Fisher [12] argued that Gresham's law(s) apply in domestic, but not in international, transactions because of domestic legal tender laws. Our formulation allows us to evaluate this argument. Velde, Weber, and Wright [28] create a model yielding some phenomena associated with Gresham's Law. Their analysis is not based on the presence of any legal tender provisions.

[4] Other models of commodity monies are by Cass and Yaari [3], Barro [1], Sargent and Wallace [24], and Fischer [11].

[5] On the small country interpretation, these technologies can be regarded as the exchange of metal for goods in international markets at competitive prices.

model of the physical depreciation of coins; and (c) a set of rules for minting new coins that follow the mechanism described above. After we invest in setting up these features, the main properties of our model flow easily from manipulating a profit function: 'no arbitrage' conditions force the price level to inhabit a particular interval of values, and impose sharp restrictions on the covariation of the price level with rates of minting and melting coins. After deducing these restrictions, we construct a variety of examples to display the sorts of equilibria our 'curious mechanism' might call forth. Among these are equilibria in which debasements prompt voluntary surges in recoining and seigniorage revenues, as well as equilibria that conform to – or violate – various versions of Gresham's Law and the quantity theory of money.

The paper is organized as follows. The remainder of this introduction describes some historical observations, and how various doctrines about Gresham's Law evolved to confront them. Section 2 describes a basic monometallic version of our model, and derives the key arbitrage and other equilibrium conditions. Section 3 describes a variety of sample equilibria for the monometallic model designed to be consistent with stylized versions of some of the historical patterns we report. Section 4 briefly takes up a bimetallic version of the model, while Section 5 summarizes and announces our hopes for its future applications. Before diving into our calculations, it is useful to devote a few paragraphs to various versions of Gresham's Laws and the events that inspired their formulation.

Gresham's laws and their obverses[6]

Gresham's Law occurs in various forms depending on the particular objects that are identified as bad and good money.[7] In bimetallic settings 'bad' has meant money overvalued at the mint – say gold (or silver) – and 'good' is money undervalued at the mint – say silver (or gold). In mono-metallic settings, 'bad' has been taken to be under-weight coins and 'good' to be full-weight coins. Such a taxonomy of Gresham's Laws was presented by Fisher [12] and Jevons [17]. Fisher traced scientific formulations of Gresham's Law back to Bishop Oresme and Copernicus, and recounted a clear statement of an under-weight coins version in Aristophanes' *Frogs*.

To compound confusion, versions of Gresham's Law have sometimes and somewhere seemed to apply, but at other times and elsewhere to be contradicted. Thus, bimetallic monetary systems often went through periods when only one type of money was coined and the other was 'shipped

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[6] The American Heritage Dictionary defines the noun obverse as: 1. The side of a coin or medal that bears the principal stamp or design. 2. A counterpart or complement.

[7] See de Roover ([9], especially page 93) for an account of Thomas Gresham's own views about his Law.

out' (or melted),[8] episodes that seem to confirm a version of Gresham's law.[9] But we know of other times[10] when two metals coexisted as money, to be followed by episodes in which only a single metal functioned as money. Moreover, these transitions from multiple to single metallic currencies did not always seem to be accompanied by changes in mint prices, in the way that Gresham's law would suggest. Our model economy can account for these events by displaying cycles in the monetary functions of different metals, even though the relative intrinsic (and legal) values of the different metals remain constant forever.

Within mono-metallic regimes, many periods saw under-weight coins circulating side-by-side (and at par) with coins of greater intrinsic value, contradicting another version of Gresham's law.[11, 12] This situation preoccupied Jevons [17]. Jevons deplored the fact that 'in England at the present day the force of habit ... lead[s] to the depreciation of our gold standard coinage by abrasion .... Every standard coin thus tends to degenerate into a token coin.' (p. 82) Jevons asked 'how long shall we in England allow our gold coinage to degenerate in weight?' (p. viii) Moreover, this feature of the monetary system that so concerned Jevons had persisted for some time. National concern about the underweight British coinage had engaged Locke, Lowndes, and Newton in discussions of alternative remedies shortly after the Glorious Revolution of 1688 (see Horsefield [16], pp. 49–51).

Jevons ([17], p. 110) tried to reconcile some of these conflicting observations by positing that Gresham's Law applies to full-weight, but not under-weight coins. Our model depicts some conditions under which Jevons is correct, but also different conditions under which Gresham's Law applies both to full-weight and to under-weight coins, or not at all.[13]

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[8] One example is the U.S. before the 1850s. See Carothers [2]. Cipolla ([6], pp. 33–35) describes examples in the Florentine context.

[9] Jevons ([17], p. 100) thought that under a bimetallic system 'the currency will tend to become composed alternately of one or the other metal, and money-changers will make a profit out of the conversion.' This situation has prompted an inference that bimetallic standards served no purpose, because they tended to be de facto mono-metallic standards (Redish [20]). Some have argued that the existence of a bimetallic standard can reduce the extent of price level fluctuations (Jevons ([17], p. 135–6), Friedman [13], Oppers [19].

[10] See. Cipolla ([6], p. 17) for one example. We recommend Cipolla [5] for general observations about commodity monies.

[11] This circumstance has even been observed when an over-weight coin was minted with the intention that it *should* be shipped abroad. See Rolnick and Weber's [22] account of the U.S. 'trade dollar' of the late 19th century.

[12] In 1692, John Locke described a possible equilibrium in which under-weight money 'tis easily changed into weighty Money,' and that to the holder of currency "it is all one to him whether he received his Money in clip'd Money, or no, so it be all current.' This quotation is from John Locke, *Some considerations of the Consequences of the lowering of Interest, and raising the value of Money*, 1692; quoted by J. Keith Horsefield. [16], p. 59.

[13] A new paper by Greenfield and Rockoff [15] describes a variety of episodes in 19th century U.S. monetary history which they cite as evidence of circulation by tale.

We also use our model to study the operation of the quantity theory under a commodity money (fixed exchange rate) system. According to Friedman and Schwartz ([14], pp. 89–90):

> Under a gold standard with fixed exchange rates, ... the stock of money is ultimately a dependent factor controlled primarily by external influences. ...
> The major channel of influence is from the fixed rates of exchange with other currencies through the balance of payments to the stock of money, thence to the level of internal prices that is consistent with those exchange rates.

Yet Friedman and Schwartz often discuss relative movements in real income and the stock of money during times when the U.S. was on a gold standard, and use these movements to explain the behavior of the U.S. price level. They justify such an analysis by arguing that "there was some leeway in short periods," (p. 683) so that closed economy versions of the quantity theory – whereby movements in the money supply relative to real income influence the price level – can be applied.

When do various versions of the quantity theory apply under commodity money systems? Cipolla [6] told of a string of episodes exhibiting cycles of coining, melting, and of re-coining the *same* metal in 14th century Florence, including one called 'the affair of the quattrini'. The coinings and meltings were associated with puzzling behavior of currency values, however. Thus, Cipolla (p. 48–51) described a flurry of mint activity in Florence during the years 1348–1351. Over these years, the money supply presumably expanded rapidly. Yet the value of the currency coined 'remained stable ... until 1350.' Minting ended abruptly in 1351, and currency values did not erode much until about 1355. So it seems that currency did not fall in value until the cycle of coining was complete. Similarly, Cipolla reported heavy minting of a coin called the *quattrino* over a period 1372–5. Depreciation of this coin on a systematic basis was not observed until at least 1374. Finally, in 1381, quattrini were melted in massive quantities. Yet this melting did not increase the value of the currency; according to Cipolla (p. 84) 'on the basis of the only data at our disposal, it is difficult to explain the paradox.'[14] Here Cipolla is questioning why closed economy versions of the quantity theory do not apply, as well as the timing of money and price level movements (the existence of 'long and variable lags'). Our model is consistent with Cipolla's observations, and asserts that a rise in currency values should not have been observed until the melting phase was finished. We can, therefore, explain not only the failure of the quantity theory to apply to this episode, but also the timing of price level movements relative to the behavior of the money supply.

[14] Cipolla conjectured an inflow of foreign currency, but adduces no data to support his conjecture.

That our model lets under-weight coins circulate at face value allows us to study other historical puzzles. Davis and Hughes [8] observed that dollar-sterling exchange rates rarely respected the bands implied by legal valuations of respective currencies and shipping costs in the first three quarters of the 19th century. Our analysis suggests that the importance of under-weight coins during much of this period (Carothers [2]) can potentially account for such an observation.

2 A mono-metallic economy

Preferences and technology

The economy consists of a set of identical, infinitely-lived households. In each period $t \geq 1$, a representative household is endowed with $\xi_t > 0$ units of a single consumption good. Letting c_t denote date t consumption, the household's lifetime utility is given by

$$\sum_{t=1}^{\infty} \beta^{t-1} v(c_t); \quad \beta \in (0, 1), \tag{1}$$

where v is an increasing, concave function. We impose the following Inada condition:

$$\lim_{c \searrow 0} v'(c_t) = +\infty. \tag{2}$$

All money consists of silver coins, called 'dollars', which are minted or melted at the discretion of private citizens, under rules of minting set forth by the government. Coins assume one of two conditions, 'full-weight' and 'under-weight.' Under-weight coins have a fraction $\alpha \in (0, 1)$ of the silver content of full-weight coins. Full-weight coins are produced when people bring silver to the mint. Under-weight coins are created when full-weight coins depreciate. The government sets a mint price in the form of a parameter b_s, whose units are ounces of silver per full-weight dollar. Only full-weight dollars can be minted. We let n_t denote the rate of minting of full-weight dollars; μ_t^f the rate of melting of full-weight dollars; and μ_t^u the rate of melting of under-weight dollars. Let f_t denote the stock of full-weight dollars, and u_t the stock of under-weight dollars. Each of the variables $n_t, \mu_t^f, \mu_t^u, f_t, u_t$ is measured in nominal terms. We assume the laws of motion

$$\begin{aligned} f_t &= \delta f_{t-1} + n_t - \mu_t^f \\ u_t &= u_{t-1} + (1 - \delta) f_{t-1} - \mu_t^u \end{aligned} \tag{3}$$

Thus, full-weight coins depreciate at the rate of $(1 - \delta)$, but under-weight coins don't depreciate.[15]

[15] This simplifying assumption is made to keep the state of the economy of small dimension; all coins are either full-weight or under-weight. Under constant proportional depreciation, it would be necessary to keep track of the vintage structure of the currency.

The rates of new coinage and melting are subject to the constraints

$$
\begin{aligned}
n_t &\geq 0 \\
\delta f_{t-1} &\geq \mu_t^f \geq 0 \\
u_{t-1} + (1-\delta) f_{t-1} &\geq \mu_t^u \geq 0.
\end{aligned} \tag{4}
$$

We assume that coins circulate according to their face values, so that 'a dollar is a dollar.' Then the money supply carried from t to $t+1$ is

$$m_t = f_t + u_t. \tag{5}$$

Equations (3) and (5) give the law of motion for the money supply

$$m_t = m_{t-1} + n_t - \mu_t^f - \mu_t^u. \tag{6}$$

Government

To address the historical observations on seigniorage fluctuations, we posit that the government imposes a seigniorage fee in the amount σn_t, where $\sigma \geq 0$. The government's budget constraint at time t is

$$\sigma n_t = p_t g_t,$$

where g_t is government purchases of goods at t. Because n_t is endogenous, so are the amounts of seigniorage and government purchases.

Feasibility

There is a reversible linear technology for converting the consumption good into silver: it yields ϕ_s ounces of silver per unit of the consumption good.[16] For the present, we assume that silver in the form of coins is the only commodity that can be carried between periods.[17] The ratio $\gamma = \phi_s / b_s$ describes the number of dollars to be obtained by converting a unit of the consumption good into silver, and then having the silver coined. In addition, there is an irreversible technology for converting under-weight coins into consumption goods. Upon being melted, an under-weight dollar yields $\gamma^{-1}\alpha$ units of the consumption good.[18]

There is an exogenous endowment sequence $\{\xi_t\}_{t=1}^{\infty}$ of the consumption good, which together with the proceeds from melting coins, can be allocated between consumption and new coinage. The resource constraint is

$$c_t + g_t + \gamma^{-1} n_t \leq \xi_t + \gamma^{-1} \mu_t^f + \alpha \gamma^{-1} \mu_t^u. \tag{7}$$

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[16] On a 'small country' interpretation, ϕ_s is the world price of consumption goods relative to silver.

[17] Later we shall describe a version of the model in which raw silver can be stored too.

[18] We have just described a closed economy. We can also interpret it as a small open economy that can import or export specie in exchange for commodities at a constant world price. Under this interpretation, coins pass according to their face value in domestic transactions and according to weight in international transactions. Fisher [12] and Jevons [17] assert that this was the typical practice when they wrote. Fisher attributed this situation to the force of domestic legal tender laws. Also see Smith ([27] Book 1, chapter 5).

Within period timing

We adapt the usual (see Lucas [18]) shopper-worker decomposition of the household to support the following within-period timing of events at t. We use a small country interpretation of the 'technology' for transforming consumption goods from and into silver. First, a household-owned firm receives consumption goods in the form of an endowment ξ_t and net imports of $\gamma^{-1}(\mu_t^f + \alpha\mu_t^u - n_t)$. Second, the shopper uses the stock of money m_{t-1} carried over from $t-1$, which at the beginning of t is composed of δf_{t-1} full-weight dollars and $u_{t-1} + (1-\delta)f_{t-1}$ underweight dollars, to purchase consumption goods from the firm. The firm must accept under-weight dollars at par with full-weight dollars, and cannot discriminate between the two in transactions. Third, the firm melts and mints dollars at rates μ_t^u, μ_t^f, n_t, and uses the net proceeds of metal to settle its import account. The firm pays taxes of σn_t dollars to the government, which immediately uses all of the proceeds to purchase g_t units of the consumption good from the firm.[19] These operations leave the firm holding no consumption goods and stocks of f_t full-weight and u_t under-weight dollars. Fourth and finally, the firm pays its net nominal proceeds $m_t = f_t + u_t$ to the 'worker.' The firm carries neither money nor goods between periods. The household carries money between periods and arrives at the start of period $t+1$ with $u_t + (1-\delta)f_t$ under-weight coins and δf_t full-weight coins.

We have designed our model to be sensible so long as the cash-in-advance constraint always binds, which renders the household indifferent about the composition of its money holdings between full-weight and under-weight coins. Were the constraint occasionally not to bind, the household would have reason to care about the composition of its cash holdings. It would typically prefer to hold under-weight coins, a feature that originates in the peculiar depreciation technology we have assumed, which makes under-weight coins a superior device for storing silver. Accordingly, we shall not study equilibria where the cash-in-advance constraint occasionally fails to bind.

The firm

During period t, the firm sells to the 'shopper' goods consisting of the endowment ξ_t, and 'net imports' Im_t; Im_t can be positive or negative, and satisfies

$$Im_t = \gamma^{-1}(\mu_t^f + \alpha\mu_t^u - n_t).$$

The firm acquires net-imports in exchange for silver - at the 'world' rate of exchange of silver for goods γ. The firm chooses (n_t, μ_t^u, μ_t^f) to maximize its one-period profits $\Pi_t = p_t(\xi_1 + Im_t) + (n_t - \mu_t^f - \mu_t^u) - \sigma n_t$, where the firm pays the seigniorage fee σn_t when $n_t > 0$. Using the definition of Im_t, we can rewrite the firm's profits as

[19] Notice that the cash-in-advance constraint applies neither to the government nor to 'importers'.

$$\Pi_t = p_t\xi_t + (p_t\gamma^{-1} - 1)\mu_t^f + (p_t\alpha\gamma^{-1} - 1)\mu_t^u + (1 - p_t\gamma^{-1} - \sigma)n_t. \tag{8}$$

The firm maximizes (8) subject to the non-negativity constraints (4). Constraints (4), along with the 'no-arbitrage principle' that in any equilibrium the right-hand side of (8) must be bounded, implies the following equilibrium conditions:

$$n_t \geq 0; \; = \text{if} \;\; p_t > \gamma(1-\sigma) \tag{9a}$$

$$\mu_t^f \leq \delta f_{t-1}; \; = \text{if} \;\; p_t > \gamma \tag{9b}$$

$$\mu_t^f \geq 0; \; = \text{if} \;\; p_t < \gamma \tag{9c}$$

$$\mu_t^u \geq 0; \; = \text{if} \;\; p_t < \gamma/\alpha \tag{9d}$$

$$(1-\delta)f_{t-1} + u_{t-1} \geq \mu_t^u; \; = \text{if} \;\; p_t > \frac{\gamma}{\alpha}. \tag{9e}$$

These conditions contain much of the information that we shall use to bound the price level, and to characterize the covariation of the price level with minting and melting decisions.

At the end of the period, the firm dispenses it profits Π_t to the household in the form of coins.

The household

Given $m_0, \{\Pi_t\}_{t=1}^{\infty}$, the household maximizes (1) subject to the cash-in-advance constraint[20]

$$p_t c_t \leq m_{t-1}, \tag{10}$$

and the sequence of budget constraints

$$m_t = \Pi_t + (m_{t-1} - p_t c_t) \quad \forall\, t \geq 1. \tag{11}$$

The household's problem induces the following 'Euler inequality':

$$\frac{\beta v'(c_t + 1)}{p_t + 1} \leq v'(c_t)/p_t; \quad \text{and} \quad m_{t-1} \geq p_t c_t, \tag{12}$$

with at least one equality obtaining. We focus on situations where the cash-in-advance constraint is binding, meaning that the first inequality of (12) holds strictly.

The Inada condition implies that $c_t > 0$, which in conjunction with (12) implies that $m_t > 0$ in any equilibrium. This will imply $u_t > 0$, which with (9e) implies that $p_t \leq \frac{\gamma}{\alpha}$. It then follows from (9) that the price level satisfies

$$(1-\sigma)\gamma \leq p_t \leq \gamma/\alpha. \tag{13}$$

In a small open economy, (13) restricts the price level to respect specie import and export points. The inequalities in (9) limit the ways in which minting and melting (or the importation and exportation of specie) can covary with prices.

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[20] This constraint embodies the timing conventions of Cooley and Hansen [7].

Equilibrium

We use

Definition: Given a sequence $\{\xi_t\}$ and a pair of values (ϕ_s, b_s), an *equilibrium* is a collection of sequences that satisfy equations (3)–(12).

This definition incorporates both market clearing and individual optimization.

The laws of motion for the stocks of coins (3) and the relations between the price level and minting or melting activity (9) have several interesting implications for the equilibrium outcomes that can be observed. The two sets of relations by themselves delineate the following possibilities:

(a) Full-weight coins will be minted only if $p_t = \gamma(1 - \sigma)$.
(b) If $p_t \in ((1 - \sigma)\gamma, \gamma)$, no minting occurs but full-weight and under-weight coins coexist, and no coins are melted.
(c) If $p_t > \gamma$, all full-weight coins are melted $(\mu_t^f = \delta f_{t-1})$. If $\gamma < p_t < \gamma/\alpha$, no under-weight coins are melted.
(d) If $p_t = \gamma/\alpha$, some under-weight coins might be melted.

The price level has a floor $\gamma(1 - \sigma)$, determined by the goods value of newly minted coins and the seigniorage rate. It is profitable to mint coins if the price level ever threatens to go below this level. The price level also has a ceiling of γ/α, enforced by the profitability of melting under-weight coins if the price level were to rise above this value. There is room for the price level to fluctuate within the bands $\gamma(1 - \sigma) \leq p_t \leq \frac{\gamma}{\alpha}$, so long as the quantities of consumption, melting, and coinage covary with the price process in ways that respect the Kuhn-Tucker conditions (9). With $\alpha < 1$ or $\sigma > 0$, there is also room for a closed economy version of the quantity theory to operate; in particular, if the cash-in-advance constraint is binding and $p_t \in (\gamma(1 - \sigma), \gamma/\alpha)$, a quantity-theory relation holds, with the price level being proportional to the quantity of money brought into the period. [21]

3 Sample equilibria

Various kinds of equilibria can be observed when coins pass by their face values. To illustrate some of the possibilities, we construct several types of equilibria.

[21] Adam Smith ([27] Book 1, chapter 5) observed that a seigniorage fee would permit the value of coins to exceed their extrinsic values as metals. To explain an excess value as coins over metal, he also alluded to delivery delays at the mint which lifted the value of newly minted coins by a factor to compensate for foregone interest. Smith interpreted an excesss value at home (where coins circulated by tale) relative to abroad (where coins were valued by weight) as a force propelling the eventual return of full-weight and under-weight coins that had been shipped abroad in times of domestic stringency. Also see David Ricardo ([21], pp. 238–239).

Our strategy for constructing equilibria is to guess a price level sequence satisfying (13), then to find restrictions on endowment sequences such that equations (3)–(9) and (12) are satisfied.[22] When $\sigma > 0$, we construct the following types of equilibria.

1. An 'anti-Gresham's Law' equilibrium in which $p_t = (1 - \sigma)\gamma$ for all t, coins are never melted, and full-weight and under-weight coins coexist.
2. An equilibrium that displays features of the (closed economy version of the) quantity theory of money, in which $p_t \in ((1 - \sigma)\gamma, \gamma)$, for all t. Gresham's Law does not hold.
3. An equilibrium in which $p_t \in ((1 - \sigma)\gamma, \gamma]$, and in which $p_t = \gamma$ for one date t_0. In this equilibrium, full weight coins are melted only at date t_0. Under-weight coins are never melted.
4. An equilibrium in which $p_t = \gamma$, the economy is never growing, and the stock of full-weight coins is gradually melted.
5. An equilibrium that displays features of both Gresham's Law and the (closed economy version of the) quantity theory of money, in which $p_t \in (\gamma, \gamma/\alpha)$ for all t.
6. An equilibrium in which $p_t = \gamma/\alpha$ for all t, which displays features of Gresham's Law.
7. A patching together of two equilibria designed to represent a hypothetical 'debasement experiment'. A carefully crafted debasement shifts an initial equilibrium of type 6 into an equilibrium of type 1, leaves the price level and consumption allocation unchanged, and raises revenues for the government.

We also consider some equilibria in which $\sigma = 0$.

8. An 'anti-Gresham's Law' equilibrium in which $p_t = \gamma$ for all t, and full-weight and under-weight coins coexist.
9. A periodic equilibrium in which the price level fluctuates between γ and γ/α, with associated cycles of coining and melting.
10. An equilibrium with a temporary period of high price levels and sustained melting of under-weight coins, generated by a declining endowment.

Some calculations and proofs are relegated to the appendix.

[22] By starting with what is ordinarily regarded as a sequence of endogenous variables-prices-and finding sequences of the exogenous and other endogenous variables that can support it in an equilibrium, we are using a version of Sims's [25, 26] 'back-solving' algorithm. Another version of this algorithm underlies the calculations of Diaz-Giménez, Prescott, Fitzgerald. and Alvarez [10].

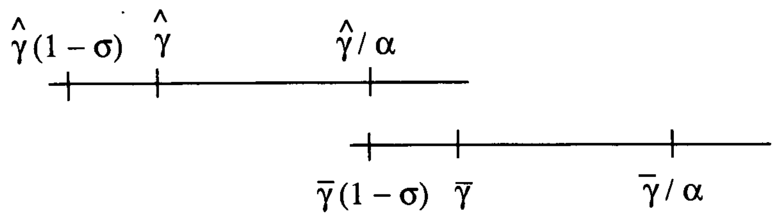

Figure 1. A noninflationary debasement raises γ from an initial value $\hat{\gamma}$ to a terminal value $\bar{\gamma}$ satisfying $\bar{\gamma}(1-\sigma) = \hat{\gamma}\alpha$. This debasement accompanies a price level that is constant at $p_t = \hat{\gamma}/\alpha$, and raises revenue for the government.

Type 1 equilibrium: perpetual minting

In such an equilibrium, endowment growth causes continued growth in consumption and the money supply. The price level is stuck at its lower bound. This equilibrium has[23]

$$
\begin{aligned}
p_t &= \gamma(1-\sigma) \\
\mu_t^u &= \mu_t^f = 0 \\
c_t &= \xi_{t-1}, \; t \geq 2 \\
c_1 &= \frac{m_0}{\gamma(1-\sigma)} \\
n_t &= \gamma(1-\sigma)(\xi_t - c_t) \geq 0, \; t \geq 1 \\
p_t g_t &= \sigma n_t \geq 0, \quad t \geq 1.
\end{aligned}
$$

Sufficient conditions for such an equilibrium to exist are that $(m_0, \{\xi_t\}_{t=1}^{\infty})$ satisfy

$$
\begin{aligned}
\beta v'(\xi_t) &< v'(\xi_{t-1}), \quad t \geq 2 \\
\beta v'(\xi_1) &< v'\left(\frac{m_0}{\gamma(1-\sigma)}\right).
\end{aligned}
$$

In such an equilibrium

$$c_{t+1} - c_t = \xi_t - \xi_{t-1} = \frac{1}{\gamma(1-\sigma)} n_t \geq 0, \;\; t \geq 2.$$

Growth necessitates perpetual minting and generates real government purchases of

$$g_t = \sigma(\xi_t - \xi_{t-1}) \geq 0.$$

This is an 'anti-Gresham's Law' equilibrium because full-weight coins and under-weight coins circulate and have identical market values.

[23] Equation (7) and the government budget constraint imply that $n_t = \gamma(\xi_t - c_t - \sigma n_t/p_t) = \gamma(\xi_t - c_t) - [\frac{\sigma}{1-\sigma}]n_t$. Solving this equation for n_t gives the equation we report for n_t.

Type 2 equilibrium: 'quantity theory' with no melting or minting

When $\sigma > 0$, there is room to construct other 'anti-Gresham's Law' equilibria in which neither melting nor minting of coins ever occurs. Such equilibria have

$$\begin{aligned} p_t &\in ((1-\sigma)\gamma, \gamma) \\ \mu_t^u &= \mu_t^f = n_t = 0 \\ c_t &= \xi_t \\ m_t &= m_0 \quad \forall\, t \geq 1 \\ p_t &= \frac{m_0}{\xi_t}. \end{aligned}$$

Evidently, such an equilibrium exists if $(m_0, \{\xi_t\}_{t=1}^{\infty})$ satisfy $\beta v'(\xi_{t+1})\xi_{t+1} < v'(\xi_t)\xi_t$ and $\gamma(1-\sigma) < \frac{m_0}{\xi_t} < \gamma$. Full-weight and under-weight coins coexist, and bear values that exceed their intrinsic values as metals.

Type 3 equilibrium: melting of full weight coins at date t_0 only

Here is an anti-Gresham's Law equilibrium in which a more or less permanent reduction endowment at t_0 causes a once and for all drop in the money supply as full-weight but not under-weight coins are melted. Suppose that $m_0, \{\xi_t\}_{t=1}^{\infty}, \gamma$, and σ satisfy all of the conditions implying existence of a type 2 equilibrium for all t *except* t_0. Suppose further that $m_0/\xi_{t_0} > \gamma$. Then there is an equilibrium with

$$p_t = \frac{m_t}{\xi_t} \in (\gamma(1-\sigma), \gamma) \quad t \neq t_0$$

$$p_{t_0} = \gamma$$

$$\begin{aligned} g_t = n_t = \mu_t^u &= \mu_t^f = 0, \quad t \neq t_0 \\ m_t &= m_0 \qquad \text{for } t \leq t_0 - 1 \\ \gamma c_{t_0} &= m_{t_0} \end{aligned}$$

$$0 < \mu_{t_0}^f = \gamma(c_{t_0} - \xi_{t_0}) < \delta f_{t-1} \tag{14}$$

$$m_{t_0} = m_0 - \gamma^{-1}\mu_{t_0}^f$$

$$m_t = m_{t_0} \quad t \geq t_0 \tag{15}$$

$$c_t = \xi_t, \quad t \neq t_0.$$

All equilibrium conditions are satisfied if

$$\begin{aligned} \mu_{t_0}^f = m_0 - \gamma\xi_{t_0} &< \delta^{t_0} f_0 \\ m_0 v'(m_0/\gamma) &< \gamma\xi_{t_0-1} v'(\xi_{t_0-1}) \\ \gamma\xi_{t_0+1} v'(\xi_{t_0+1}) &< (m_0 - \mu_{t_0}^f) v'(m_0/\gamma) \end{aligned}$$

and

$$\gamma(1-\sigma) < (m_0 - \mu_{t_0}^f)/\xi_t < \gamma; \quad t > t_0$$

hold.

Type 4 equilibrium: $p_t = \gamma$ with melting of full weight coins

In this equilibrium, $p_t = \gamma, n_t = \mu_t^u = g_t = 0$. The household's budget constraint implies $m_t = \gamma\xi_t$, and the cash-in-advance constraint at equality implies $m_t = \gamma c_{t+1}$. Therefore, $c_{t+1} = \xi_t \;\forall\, t \geq 1$ and $c_1 = \gamma^{-1}m_0$. Feasibility implies $c_{t+1} = \xi_{t+1} + \gamma^{-1}\mu_{t+1}^f \;\forall\, t \geq 1$, and so we must have $\delta f_{t-1} \geq \gamma\left[\xi_t - \xi_{t+1}\right] \geq 0, \;\forall\, t \geq 1$. We also require $\delta f_0 \geq m_0 - \gamma\xi_1 = \mu_1^f \geq 0$. Evidently, existence of such an equilibrium requires that $f_0, \{\xi_t\}_{t=1}^{\infty}$, and γ satisfy $\delta f_{t-1} \geq \gamma\left[\xi_{t-1} - \xi_t\right] \geq 0 \;\forall\, t > 1$ and $\delta f_0 \geq m_0 - \gamma\xi_1 \geq 0$. The former condition, in turn, can be reduced to

$$f_0 \geq \gamma \sum_{i=0}^{t-1} \delta^{i-t}(\xi_{t-i} - \xi_{t-i+1}) \geq 0; \quad t > 1.$$

Finally, the cash-in-advance constraint is binding at all dates if $\beta v'(\xi_{t+1}) < v'(\xi_t)$ and $\beta v'(\xi_1) < v'(m_0/\gamma)$ are satisfied. These conditions clearly require that the endowment never increase, and that it must asymptotically shrink at a rate which is not too rapid. Thus, only declining economies can experience a perpetually shrinking stock of commodity money. The decline must not be too rapid if inflation is to be avoided.

Economies satisfying the conditions of a type 4 equilibrium display a 'qualified version' of Gresham's Law: 'good' money is driven out, but so slowly as never to be completely displaced.

Type 5 equilibrium: Gresham's law with monetary fiat components

In this Gresham's Law equilibrium, 'bad' (under-weight) money drives out 'good' (full-weight) money: for $t \geq 2$, only under-weight coins are in circulation. Here there are two possibilities; we describe one now and the other in the next section. In this section, we focus on equilibrium with

$$p_t \in (\gamma, \gamma/\alpha); \; t \geq 1. \tag{16}$$

Only under-weight coins circulate after $t = 1$. These coins have a 'fiat component,' in that their monetary value exceeds their intrinsic value.

When (16) holds, we have $\mu_1^f = \delta f_0, \mu_t^u = 0 \;\; \forall\, t \geq 1, \; n_t = 0 \;\; \forall\, t \geq 1$, and $\mu_t^f = 0 \;\; \forall\, t > 1$. Hence $m_t = m_1 = u_0 + (1-\delta)f_0, \;\; \forall\, t \geq 1$.

Proposition 1. There exists an equilibrium satisfying (10) at equality and (16) $\forall\, t \geq 1$ iff

$$\gamma\xi_t/\left[u_o + (1-\delta)f_0\right] < 1 < (\gamma/\alpha)\xi_t/\left[u_0 + (1-\delta)f_0\right]; \; t \geq 2, \tag{17a}$$

$$1 < \gamma^{-1}m_0/\left[\xi_1 + \delta\gamma^{-1}f_0\right] < 1/\alpha, \tag{17b}$$

and

$$\xi_t v'(\xi_t) > \beta\xi_{t+1}v'(\xi_{t+1}); \; t \geq 1, \tag{18a}$$

$$v'\left[\xi_1 + \delta\gamma^{-1}f_0\right]\left[\xi_1 + \delta\gamma^{-1}f_0\right]/m_0 > \beta\xi_2 v'(\xi_2)/\left[u_0 + (1-\delta)f_0\right] \tag{18b}$$

hold. If such an equilibrium exists it has

$$c_1 = \xi_1 + \delta\gamma^{-1} f_0, \tag{19a}$$

$$c_t = \xi_t \ ; \ t \geq 2, \ . \tag{19b}$$

$$p_1 = m_0 / \left[\xi_1 + \delta\gamma^{-1} f_0\right], \tag{20a}$$

$$p_2 = \left[u_0 + (1-\delta) f_0\right] / \xi_2, \tag{20b}$$

$$p_{t+1}/p_t = c_t / c_{t+1} \ ; \ t \geq 2. \tag{20c}$$

The date $t = 1$ version of equation (17) requires that

$$\gamma\xi_1 + \delta f_0 < m_0 \tag{21a}$$

$$\gamma\xi_1 + \delta f_0 > \alpha m_0. \tag{21b}$$

Rewrite (21) as

$$\gamma\xi_1 < (1-\delta) f_0 + u_0 \tag{22a}$$

$$\gamma\xi_1 > (\alpha - \delta) f_0 + \alpha u_0. \tag{22b}$$

Proposition 1 describes an economy in which Gresham's Law operates. 'Bad' money drives out 'good' money, but 'bad' money circulates at a value exceeding its intrinsic value. Proposition 1 also states conditions under which a commodity money system will display a 'closed economy version' of the quantity theory. In particular, economies with type 5 equilibria have a constant money supply (after the first period), and prices that fluctuate to reflect movements in income.

In order for a type 5 equilibrium to exist, endowments must fluctuate within a range defined by equation (17a). The tightness of this range depends on how badly depreciated under-weight coins are (that is, on the size of α). The larger is α, the less likely is it that any economy can obey both Gresham's Law and this particular 'closed economy' version of the quantity theory (see, for example, equations (17a) and (21)). Thus, α governs the 'leeway' in the quantity theory referred to by Friedman and Schwartz. For small enough values of α, this leeway can be ample.

Type 6 equilibrium: 'Gresham's law' without a fiat component of value

In equilibria of this type, under-weight coins 'drive out' full-weight coins, and the value of under-weight coins is simply their intrinsic value. In these equilibria, commodity money lacks a fiat component.

For under-weight coins to circulate at their intrinsic value, it must be that

$$p_t = \gamma/\alpha \ ; \ t \geq 1. \tag{23}$$

Then clearly $\mu_1^f = \delta f_0, n_t = 0 \ \forall t \geq 1$, and $\mu_t^f = 0 \ \forall t \geq 2$ all must hold. As before, we consider only equilibria where the cash-in-advance constraint binds at each date.

Proposition 2. There exists an equilibrium in which (23) and (10) at equality hold at each date iff

$$\xi_{t-1} \geq \xi_t \ ; \ t \geq 2, \tag{24a}$$

$$v'(\alpha\gamma^{-1}m_0) > \beta v'(\xi_1) \tag{24b}$$

$$v'(\xi_t) > \beta v'(\xi_{t+1}) \; ; \; t \geq 1, \tag{25}$$

and

$$(\alpha - \delta)f_0 + \alpha u_0 \geq \gamma\xi_1. \tag{26}$$

If such an equlibrium exists, it has

$$c_1 = (\alpha\gamma^{-1})m_0 \tag{27a}$$

$$c_{t+1} = \xi_t \; ; \; t \geq 1. \tag{27b}$$

Proposition 2 asserts a sense in which only relatively poor, stagnant economies can support Gresham's Law equilibria with no fiat component. Condition (26) requires that ξ_t not be too large, and (24) requires that the endowment not grow.

In a type 6 equilibrium, Gresham's Law obtains along with an 'open economy version' of the quantity theory; the exchange rate between goods and silver is effectively fixed, and the money supply adjusts endogenously to changes in income. However, since only melting (exporting) occurs, such an equilibrium is inconsistent with endowment growth.

Debasement

We model a debasement as a 'surprise' once-and-for-all decrease in b_s at time $t_0 > 1$, leading to an increase in γ from an initial value $\hat{\gamma}$ to $\bar{\gamma}$. We let $(\hat{f}_t, \hat{u}_t), (\bar{f}_t, \bar{u}_t)$ be the stocks of old and new full-weight and under-weight coins, respectively, for $t \geq t_0$. The money supply for $t \geq t_0$ is

$$m_t \equiv \hat{f}_t + \hat{u}_t + \bar{f}_t + \bar{u}_t.$$

For $t \geq t_0$, only new full weight coins can be minted. We assume that for $t \leq t_0 - 1, \bar{f}_t = \bar{u}_t = 0$. We will confine ourselves to a particular example in which $\hat{f}_t = 0$ and $\hat{u}_t \geq 0$ for all $t \geq t_0$. If $\hat{u}_t > 0$ holds for $t \geq t_0$, then not all old coins are melted down as a result of the debasement.

Type 7 equilibrium: a noninflationary debasement

We now construct an example of a debasement at t_0 which potentially raises seigniorage revenue, but that results in no inflation. In order to construct such an example, we assume that the economy is in an initial equilibrium with $\hat{p}_t = \hat{\gamma}/\alpha$, $n_t = \mu_t^f = \mu_t^u = g_t = 0$, $c_t = \xi_t = \xi$, and $m_t = (\hat{\gamma}/\alpha)\xi \; \forall \, t \geq t_0 - 1$.[24] At $t = t_0$, the government engineers a surprise debasement, setting $\bar{b}_s < \hat{b}_s$, and choosing $\bar{b}_s$ so that $\bar{\gamma}(1 - \sigma) = \hat{\gamma}/\alpha$. We then describe conditions under which there is a post-debasement equilibrium with $m_t = (\hat{\gamma}/\alpha)\xi, c_t = \xi$, and $p_t = \bar{\gamma}(1 - \sigma) = \hat{\gamma}/\alpha \; \forall \, t \geq t_0$. Thus, consumption, the price level, and the total money stock are unaffected by the debasement. Nevertheless, as we will

[24] We have already encountered some examples that would produce this situation.

show, there can be some recoinage, and the government can raise revenue via the debasement.

Since $\hat{p}_{t_0-1} = \gamma/\alpha$, clearly $f_{t_0-1} = 0$. Let $\hat{\mu}^u_{t_0}$ be the quantity of underweight coins melted at t_0; in order for the money supply to be unchanged we must have $n_{t_0} = \hat{\mu}^u_{t_0}$. Moreover, equations (3) and (7), plus the fact that $f_{t_0-1} = 0$, imply that

$$n_{t_0} = -\bar{\gamma} g_{t_0} + \alpha(\bar{\gamma}/\hat{\gamma})\hat{\mu}^u_{t_0}.$$

Since $g_{t_0} = \sigma n_{t_0}/p_{t_0} = \frac{\sigma\hat{\mu}^u_{t_0}}{\bar{\gamma}(1-\sigma)}$, we then have that

$$\hat{\mu}^u_{t_0} = -\left[\frac{\sigma}{1-\sigma}\right]\hat{\mu}^u_{t_0} + \alpha(\bar{\gamma}/\hat{\gamma})\hat{\mu}^u_{t_0}. \tag{28}$$

Clearly, satisfaction of (28) requires that $\mu^u_{t_0} = 0$ or

$$(1-\sigma) = \frac{\hat{\gamma}}{\alpha\bar{\gamma}}. \tag{29}$$

But (29) is exactly the condition under which $\bar{\gamma}(1-\sigma) = \hat{\gamma}/\alpha$.

Clearly, then, the quantity of recoinage in this example is indeterminate; it can lie anywhere in the interval $[0, (\hat{\gamma}/\alpha)\xi]$ at $t = t_0$. Parenthetically, the same statement is true at all subsequent dates. Thus, not only is the total amount of recoinage indeterminate; its timing is as well. Of course, the quantity and timing of recoinages here affect *no* aspect of an equilibrium, except for the sequence $\{g_t\}_{t=t_0}^{\infty}$.

Rolnick, Velde, and Weber [23] pose as a puzzle that various historical debasements resulted in substantially different volumes of recoinage and seigniorage revenue for the government. As this example illustrates, our model can produce this pattern as an equilibrium outcome.

Horsefield ([16], pp. 48–50) describes a recommendation for recoinage in a Report to Parliament by William Lowndes that conforms to a Type 7 experiment. Lowndes calculated in 1695 that although under-weight silver coins weighed about 51% of full-weight coins, they traded at 80% of the value of full-weight coins. According to Horsefield, Lowndes "therefore recommended a devaluation by 20%. This would stabilize the price of silver at approximately the current level .... Lowndes claimed that creditors would not be penalised by such a devaluation ..... The only exception to this general benefit would be the rare individual who had been able to contract out of losses hitherto by insisting on payment in full-bodied coins."[25]

[25] The government rejected Lowndes's recommendation, and instead accepted John Locke's advice to remint at government expense coins accepted at tale in payment of taxes. See Horsefield ([16], pp. 61–62). Horsefield ([16], p. 52) also has an interesting account of Isaac Newton's reasons for supporting Lowndes's proposal.

Equilibria with $\sigma = 0$

Type 8 equilibrium: 'anti-Gresham's law'

In this 'anti-Gresham's Law' equilibrium, full-weight and under-weight coins coexist, and have identical market values. Thus 'bad money' (under-weight coinage) fails to drive out 'good money' (full-weight coinage). Under-weight coins have a 'fiat component' because their monetary value exceeds their intrinsic value.

In equilibria of this type,

$$p_t = \gamma \ ; \ t \geq 1 \tag{30}$$

must hold, as must $\mu_t^u + t = 0$. Proposition 3 describes other features of such an equilibrium.

Proposition 3. Suppose that (30) holds and that the cash-in-advance constraint binds at each date. An anti-Gresham's Law equilibrium exists iff

$$\gamma\xi_1 \geq (1 - \delta)f_0 + u_0, \tag{31}$$

$$\gamma\xi_t \geq \gamma(1 - \delta)\sum_{i=0}^{t-2} \delta^i \xi_{t-1-i} + \delta^{t-1}u_1 \ ; \ t \geq 2, \tag{32}$$

$$v'(\xi_t) > \beta v'(\xi_{t+1}) \ ; \ t \geq 1, \tag{33}$$

and

$$v'\left[\gamma^{-1}m_0\right] > \beta v'(\xi_1) \tag{34}$$

hold. If such an equilibrium exists, it has $c_1 = \gamma^{-1}m_0$ and

$$c_{t+1} = \xi_t \ ; \ t \geq 1. \tag{35}$$

Proposition 3 describes some conditions under which Gresham's Law fails: under-weight coins do not 'drive out' full-weight coins at any date. In addition, Friedman and Schwartz's [14] 'small open economy version' of the quantity theory is satisfied; the exchange rate between dollars and (newly produced) silver is 'fixed,' and $m_t = \gamma\xi_t$, $\forall\, t \geq 1$. Thus the money supply adjusts endogenously to variations in income in order to maintain the 'fixed' exchange rate.

For these features to be observed in equilibrium, equations (32) and (33) require that income not decline too much between any two periods. Thus, proposition 3 rules out rapid declines in income if a fixed exchange rate version of the quantity theory is to be observed and Gresham's Law to be violated.

Equations (31) and (22a) imply that no economy can have both a type 8 and a type 5 equilibrium. Also no economy can have *both* a type 8 *and* a type 6 equilibrium. This follows immediately from equations (31), (26), and $\alpha < 1$. Similarly, no economy can have *both* a type 5 *and* a type 6 equilibrium. This fact follows from a comparison of equations (22) and (26).

Type 9 equilibrium: 'gold points' and 'the affair of the quattrini'

In this section, we describe an equilibrium in which the price level satisfies

$$p_t = \begin{cases} \gamma & \text{if } t \text{ is odd} \\ \gamma/\alpha & \text{if } t \text{ is even.} \end{cases} \tag{36}$$

If we interpret the 'melting' of silver as the shipping of specie abroad by a small open economy, and 'coining' as selling goods abroad in exchange for currency, then (36) describes a situation in which the domestic price level oscillates between 'specie import and export points.' Alternatively, we can retain a literal intrepretation of coining and melting. If we do so, equations (12) and (36) imply that coinage occurs in odd periods, when the price level is low. There is no coinage (and some melting) in even periods. So the price level will not rise until after an episode of coinage is completed. This is consistent with Cipolla's [6] account of 'the affair of the quattrini.'

In particular, equations (9) and (36) imply that $\mu_t^u = 0$; t odd, $n_t = 0$; t even, and $\mu_t^f = \delta f_{t-1}$; t even.

Proposition 4. Suppose there exists an equilibrium satisfying (36) and (10) at equality $\forall\, t \geq 1$. Define $x_t \equiv (\gamma/\alpha)(\alpha\xi_t - \xi_{t-1})$. Then this equilibrium has

$$f_1 = \gamma\xi_1 - (1-\delta)f_0 - \mu_0 \tag{37a}$$

$$f_t = \sum_{i=0}^{(t-3)/2} [-\delta(1-\alpha)/\alpha]^i x_{t-2i} + [-\delta(1-\alpha)/\alpha]^{(t-1)/2} f_1;\ t \text{ odd},\ t \geq 3. \tag{37b}$$

In addition

$$c_1 = \gamma^{-1} m_0 \tag{38a}$$

$$c_{t+1} = \alpha\xi_t\ ;\ t \text{ odd} \tag{38b}$$

$$c_{t+1} = (\xi_t/\alpha) + \gamma^{-1}[\delta(1-\alpha)/\alpha] f_{t-1}\ ;\ t \text{ even},\ t \geq 2. \tag{38c}$$

Such an equilibrium exists iff the values f_t given by (37) satisfy $f_t \geq 0$,

$$(\gamma/\alpha)(\alpha\xi_t - \xi_{t+1}) \geq (\delta/\alpha) f_t\ ;\ t \text{ odd},\ t > 1, \tag{39}$$

and

$$v'\left[\gamma^{-1} m_0\right] > \alpha\beta v'(\alpha\xi_1) \tag{40a}$$

$$\alpha v'(\alpha\xi_{t-1}) > \beta v'\left[(\xi_t/\alpha) + \gamma^{-1}\left[\delta(1-\alpha)/\alpha\right] f_{t-1}\right]\ ;\ t \text{ even} \tag{40b}$$

$$v'\left[(\xi_{t-1}/\alpha) + \gamma^{-1}\left[\delta(1-\alpha)/\alpha\right] f_{t-2}\right] > \alpha\beta v'(\alpha\xi_t)\ ;\ t \text{ odd},\ t > 1. \tag{40c}$$

Equation (40) is required in order for the cash-in-advance constraint to bind at each date.

Proposition 4 describes conditions under which there are periods when 'good' money is driven out, followed by periods in which it is not. In addition, the price level fluctuates between 'import and export' points in a manner driven by a combination of fluctuations in the (endogenous) money supply and the (exogenous) pattern of endowments. Equation (39) asserts that this pattern is necessitated by sufficiently large fluctuations in income.

Thus, economies with price levels that recurrently bounced between import and export points under a gold standard should, according to our model, have experienced relatively large income fluctuations.[26, 27]

We now present an example of an economy with a type 4 equilibrium, an economy with the strong seasonal typical of the U.S. in the late 19th century.

An example

Suppose that

$$\xi_t = \begin{cases} \xi_1 & \text{if } t \text{ is odd} \\ \xi_2 & \text{if } t \text{ is even} \end{cases},$$

and that $x_t \equiv x_1 \equiv (\gamma/\alpha)(\alpha\xi_1 - \xi_2) > 0$ for t odd. In addition, assume that $\delta/\alpha \leq 1$. Then $f_t \geq 0 \quad \forall\, t$ and (39) are satisfied iff

$$\gamma\xi_1 \geq (1-\delta)f_0 - u_0 \tag{41}$$

and

$$x_1 \geq \left[\delta(1-\alpha)/\alpha\right]f_1 \tag{42}$$

both hold.[28] (40b) is satisfied if

$$\alpha v'(\alpha\xi_1) > \beta v'(\xi_2/\alpha), \tag{43}$$

and (40c) holds iff

$$v'\left[(\xi_2/\alpha) + \gamma^{-1}\left[\delta(1-\alpha)/\alpha\right]\max(f_1, f_3)\right] > \alpha\beta v'(\alpha\xi_1). \tag{44}$$

Type 10 equilibrium: melting as consumption smoothing

We can construct an equilibrium of a temporarily declining economy in which, for a time, stocks of under-weight silver coins are being melted as a device to smooth consumption. The price level stays at its upper bound until the endowment process stops shrinking, after which it falls enough to dissuade consumers from further melting.

Let

$$p_t = \begin{cases} \gamma/\alpha & \text{for } t = 1, \ldots, T-1, \\ \tilde{p} \in \left(\gamma, \frac{\gamma}{\alpha}\right) & \text{for } t \geq T \end{cases}$$

---

[26] Under a closed economy interpretation, a type 9 equilibrium displays recurrent cycles of (potentially substantial) minting, followed by no minting whatsoever. Challis ([4], pp. 694–695) presents figures showing that something closely approximating this pattern occurred in minting in Great Britain even in the late 19th century.

[27] Ricardo ([21], p. 245) advocated a paper-backed-by-gold money to avoid the waste involved in recurrent meltings and mintings.

[28] Under the assumed endowment pattern, equation (39) reduces to $x_1 \geq (\delta/\alpha)f_t$; t is odd, $t > 1$. (A.14) reduces to $f_t = x_1 - \left[\delta(1-\alpha)/\alpha\right]f_{t-2}$. Since $\delta \leq \alpha$, clearly (39) is satisfied at all dates.

$$c_t = \begin{cases} \xi_t + (\alpha/\gamma)\mu^u_{t-1} > \xi_t, & \text{for } t = 1, \ldots, T-1 \\ \tilde{\xi} & \text{for } t \geq T \end{cases}$$

We work backwards from date T. Given a $\tilde{p} \in (\gamma, \gamma/\alpha)$, we set $c_t = \xi_t = \tilde{\xi}, t \geq T$ with $\tilde{\xi} > 0$ and choose μ_{T-1} to satisfy $\mu_{T-1} = m_{T-1} = \tilde{p}\tilde{\xi}$.

For date $T-1$, we set $p_t = (\gamma/\alpha)$ and select a c_{T-1} to satisfy the following two inequalities:

$$v'(c_{T-1}) > \beta \frac{(\gamma/\alpha)}{\tilde{p}} v'(\tilde{\xi}) \tag{45a}$$

and

$$\mu^u_{T-1} = u_{T-2} - u_{T-1} > 0. \tag{45b}$$

Equation (45a) implies that the cash-in-advance constraint is binding at $T-1$, while (45b) implies that melting occurs at that date. Since (10) is an equality, our assumption on the price level process implies that (45b) is equivalent to

$$(\gamma/\alpha)c_{T-1} > \tilde{p}\tilde{\gamma} = p_T c_T. \tag{46}$$

If $\beta(\gamma/\alpha)/\tilde{p} < 1$, then it is possible to select a value $c_{T-1} \geq \tilde{\xi}$ satisfying (45). We must then, of course, set $m_{T-1} = u_{T-1} = \tilde{p}\tilde{\xi}$, which requires that

$$\mu^u_{T-1} = (\gamma/\alpha)c_{T-1} - \tilde{p}\tilde{\xi}.$$

Resource feasibility then obligates us to set

$$\xi_{T-1} = c_{T-1} - (\alpha/\gamma)\mu^u_{T-1}.$$

Having obtained c_{T-1} and $u_{T-2} = u_{T-1} + \mu^U_{T-1}$ in this way, we can move backward one period and find c_{T-2} and μ^u_{T-2} in the same way. We can continue in this way, finding a sequence $\{\xi_t\}_{t=1}^{T-1}$ and a value m_0 that supports the desired equilibrium.

A digression on direct storage of silver

Suppose that we let households store non-negative amounts of raw silver, denoted $s_t \geq 0$ between t and $t+1$, and that raw silver does not depreciate. This would alter the feasibility condition (7) by adding $\phi_s^{-1}(s_t - s_{t-1})$ to the left-hand side, and the household's budget constraint (11) by adding $p_t\phi_s^{-1}(s_{t-1} - s_t)$ to the right-hand side. To the other first-order conditions of the household would be added the following Euler inequality for holding raw silver:

$$p_t \frac{v'(c_{t+1})}{p_{t+1}} \geq \beta p_{t+1} \frac{v'(c_{t+2})}{p_{t+2}}, = \text{if } s_t > 0.$$

Notice the *timing*. The workings of the cash-in-advance constraint imply that raw silver carried between t and $t+1$ permits *consumption* to be transferred from period $t+1$ to $t+2$.[29]

Apparently, $s_t = 0$ in a type 8 or type 6 equilibrium whenever the cash-in-advance constraint is binding. Alternatively, it is possible to construct equilibria of a model with silver storage in which $p_t \in (\gamma, \gamma/\alpha)\ \forall\ t$, so that no melting or minting occurs; and in which consumption smoothing is entirely effected via accumulation and decumulation of silver stocks. Such an equilibrium can be constructed by a method like that used to construct our type 5 equilibrium.

4 A bimetallic economy

Consider an economy identical in all respects to the one above, except that it is bimetallic. There are now two metals that can be coined, say gold and silver, and we assume that all coins circulate by tale. We let $\phi_s(\phi_g)$ be the quantity of ounces of silver (gold) obtained from one unit of the consumption good, and as before we assume that the process of producing each metal is reversible. The government specifies two mint parameters, b_s (b_g), stating how many ounces of silver (gold) constitute a full-weight dollar coin. We continue to assume that there are two types of coin (of each metal); full- and under-weight. An under-weight silver (gold) coin has a metallic content equal to a fraction α_s (α_g) of a full-weight coin. In each period a fraction $1-\delta_s(1-\delta_g)$ of previously full-weight silver (gold) coins become under-weight. We let $f_t^s(f_t^g)$ denote the number of full-weight silver (gold) coins in circulation at t – in each case measured in dollars – and $u_t^s(u_t^g)$ denotes the number of under-weight silver (gold) coins in circulation at t, again measured in dollars. We define $\gamma_g = \phi_g/b_g$ and $\gamma_s = \phi_s/b_s$.

We modify the government budget constraint to $\sigma(n_t^g + n_t^s) = p_t g_t$, where $\sigma \geq 0$ is the common seigniorage tax rate applied to new coinage of silver and gold coins. Since coins circulate according to their face value we have

$$m_t = f_t^s + f_t^g + u_t^s + u_t^g. \tag{47}$$

In addition let $n_t^s(n_t^g)$ be the quantity (in dollars) of newly-minted – and hence full-weight silver – (gold) coins at t, and let $\mu_t^{fs}(\mu_t^{fg})$ be the quantity of full-weight coins melted at the same date. Similarly, let $\mu_t^{us}(\mu_t^{ug})$ be the dollar value of under-weight silver (gold) coins that are melted at t. The evolution of the stock of silver and gold coins is described by

$$f_t^s = \delta_s f_{t-1}^s + n_t^s - \mu_t^{fs} \tag{48a}$$

[29] This discussion presumes that raw silver experiences no depreciation. It is straightforward to assign non-monetary silver a depreciation technology like that we have given monetary silver, with a depreciation factor $\delta_m \geq \delta$, and to attain a weight equal to a fraction $\alpha_m \geq \alpha$ of its original weight. Of course, with $\delta_m < 1$, this weakens the motives for storing raw silver.

$$u_t^s = (1-\delta_s)f_{t-1}^s + u_{t-1}^s - \mu_t^{us} \tag{48b}$$

$$f_t^g = \delta_g f_{t-1}^g + n_t^g - \mu_t^{fg} \tag{48c}$$

$$u_t^g = (1-\delta_g)f_{t-1}^g + u_{t-1}^g - \mu_t^{ug}. \tag{48d}$$

The choice problem of the household remains the same.The problem of the firm is modified: the firm chooses a list of sequences $\{n_t^s, n_t^g, \mu_t^{fs}, \mu_t^{fg}, \mu_t^{us}, \mu_t^{ug}\}$ to maximize profits now defined as

$$\begin{aligned}\Pi_t = p_t\xi_t + (1-p_t\gamma_s^{-1})(n_t^s - \mu_t^{fs})\\ +(1-p_t\gamma_g^{-1})(n_t^g - \mu_t^{fg}) + (p_t\alpha_s\gamma_s^{-1} - 1)\mu_t^{us}\\ +(p_t\alpha_g\gamma_g^{-1} - 1)\mu_t^{ug} - \sigma(n_t^s + n_t^g).\end{aligned} \tag{49}$$

The firm maximizes one-period profits subject to the constraints

$$n_t^g, n_t^s \geq 0 \tag{50a}$$

$$\mu_t^{fs} \in [0, \delta_s f_{t-1}^s],\ \ \mu_t^{fg} \in [0, \delta_g f_{t-1}^g] \tag{50b}$$

$$u_{t-1}^g + (1-\delta_g)f_{t-1}^g \geq \mu_t^{ug} \geq 0 \tag{50c}$$

$$u_{t-1}^s + (1-\delta_s)f_{t-1}^s \geq \mu_t^{us} \geq 0 \tag{50d}$$

In any equilibrium where the money stock is positive at each date, the following conditions are implied by the combination of 'no-arbitrage' requirements and (2):[30]

$$\max[(\gamma_s/\alpha_s), (\gamma_g/\alpha_g)] \geq p_t \geq (1-\sigma)\max[\gamma_s, \gamma_g];\ \ t \geq 1. \tag{51}$$

The firm sets

$$n_t^g = 0 \quad \text{if } (1-\sigma)\gamma_g < p_t \tag{52a}$$

$$n_t^s = 0 \quad \text{if } (1-\sigma)\gamma_s < p_t \tag{52b}$$

$$\mu_t^{fg} = \delta_g f_{t-1}^g \quad \text{if } \gamma_g < p_t \tag{52c}$$

$$\mu_t^{fs} = \delta_s f_{t-1}^s \quad \text{if } \gamma_s < p_t \tag{52d}$$

$$\mu_t^{ug} = 0 \quad \text{if } \gamma_g/\alpha_g > p_t \tag{52e}$$

$$\mu_t^{us} = 0 \quad \text{if } \gamma_s/\alpha_s > p_t. \tag{52f}$$

Goods market clearing requires that

$$c_t = \xi_t - \gamma_g^{-1}(n_t^g - \mu_t^{fg}) - \gamma_s^{-1}(n_t^s - \mu_t^{fs}) + \alpha_g\gamma_g^{-1}\mu_t^{ug} + \alpha_s\gamma_s^{-1}\mu_t^{us} \tag{53}$$

hold in each period.

We shall assume that $\sigma = 0$, and begin by also assuming that

$$\gamma_g \geq \gamma_s. \tag{54}$$

[30] Note that boundedness of the right side of the firm's profits imply that $p_t \geq (1-\sigma)\gamma_s$ (for otherwise the firm could reap unbounded profits by coining silver coins in unbounded amounts), and similarly $p_t \geq (1-\sigma)\gamma_g$.

Equation (54) asserts that silver is *not overvalued* at the mint. Then there are three possible configurations of parameters that govern the kinds of equilibria that can occur.

Case 1. Suppose that

$$\gamma_g/\alpha_g > \gamma_s/\alpha_s > \gamma_g \geq \gamma_s. \tag{55}$$

Then if $\gamma_g/\alpha_g \geq p_t > \gamma_s/\alpha_s$ holds at date t, the only money in circulation consists of underweight gold coins. If $\gamma_s/\alpha_s \geq p_t > \gamma_g$, there exist underweight gold and silver coins in circulation at t. There are no full-weight coins. Finally, if $p_t = \gamma_g$ there potentially are full-weight gold coins in use as money, as well as under-weight gold and silver coins.[31]

Equilibria with $p_t = \gamma_g(\neq \gamma_s)$ display one version of Gresham's Law while violating another. Because silver is 'under-valued' at the mint, it is never coined. In this sense the over-valued metal is the only one that can actively be minted. However, under-weight gold and silver coins circulate side-by-side, so that under-weight silver coins need not be displaced by gold. Jevons [17] asserted that Gresham's Law might apply to full-weight but not to under-weight coins in this way, and gave some examples that he believed illustrated this possibility.

We can construct economies that possess equilibria with each of these configurations at some date. For example, the construction of equilibria with $p_t = \gamma_g \ \forall\, t \geq 1$ mimics the construction of type 8 equilibria in section 3 (except that full-weight silver coins are melted at $t = 1$). It is also possible to produce economies in which p_t and the composition of the money supply vary through time. The construction of such economies mirrors the construction of type 4 equilibria in section 3.

Case 2. It is also possible that

$$\gamma_g/\alpha_g > \gamma_g \geq \gamma_s/\alpha_s > \gamma_s. \tag{56}$$

If $\gamma_g > \gamma_s/\alpha_s$ holds, then $p_t \in [\gamma_g, \gamma_g/\alpha_g] \ \forall\, t \geq 1$. After period 1 only gold coins can circulate in this economy. Here, in contrast to Jevons (1918), Gresham's Law applies both to full-weight and under-weight coins.

Case 3. Suppose that

$$\gamma_s/\alpha_s > \gamma_g/\alpha_g > \gamma_g \geq \gamma_s. \tag{57}$$

If $\gamma_s/\alpha_s \geq p_t > \gamma_g/\alpha_g$, then only under-weight silver coins circulate at t. If $\gamma_g/\alpha_g \geq p_t > \gamma_g$, under-weight gold and silver coins can circulate at t, although there can be no full-weight coins at that date. The existence of any full-weight coins at t requires $p_t = \gamma_g$.

It is possible, under (57), to produce economies with

$$p_1 = \gamma_s/\alpha_s \tag{58a}$$

[31] There can be no full-weight silver coins in circulation unless $\gamma_s = \gamma_g$.

$$p_t = \gamma_g;\ t \ \text{even} \tag{58b}$$

$$p_t = \gamma_g/\rho;\ t \ \text{odd}, \tag{58c}$$

where ρ is a constant satisfying

$$\alpha_g > \rho > \alpha_s\gamma_g/\gamma_s. \tag{59}$$

We refer to this as a 'type 11' equilibrium. Type 11 equilibria display periods (here *even* periods) where gold is coined and, in subsequent periods, all the remaining full-weight coins are melted. By construction, all coining and melting is confined to one metal. Such equilibria possess two of the properties discussed by Cipolla [6]: inflation does not occur until after recoinages are 'complete', and an economy can go through cycles of recoinage that affect only a subset of the coins in existence. In the appendix, we construct an economy with a type 11 equilibrium.

5 Conclusions

Study of commodity money systems entails several issues that do not arise in the analysis of a fiat money system. One must consider how monetizing an object with some intrinsic value affects its market value. In addition, there are issues concerning depreciation, token coinages, and whether coins circulate according to weight or tale. In bimetallic systems, the relationship between mint ratios and the intrinsic (or market) values of various metals must be considered, and Gresham's Law lurks as an issue to be confronted.

We have shown that monetizing an object may or may not raise its market value above its intrinsic value, and have described some conditions under which Gresham's Law will operate as well as conditions under which it will not.

Other topics also deserve more attention. One is the operation of an international commodity money regime. A second is the possibility of convertible paper currencies, and a third is the effect of commodity money systems in economies with production.

Appendix

A. Proof of Proposition 1

Since $n_t = \mu_t^f = \mu_t^u = 0,\ \forall\, t > 1$, equation (7) implies that $c_t = \xi_t; t > 1$. In addition, $n_1 = \mu_1^u = 0$ and $\mu_1^f = \delta f_0$ hold. Hence (7) also implies that $c_1 = \xi_1 + \delta\gamma^{-1} f_0$.

Furthermore, since $m_t = m_1 = u_0 + (1-\delta)f_0\ \forall\, t \geq 1$, (10) at equality implies that $p_t c_t = p_{t+1}c_{t+1}\ \forall\, t \geq 2$. Thus (20c) holds. At $t = 2$ we have $p_2 c_2 = p_2 \xi_2 = m_1$, delivering (20b). At $t = 1, p_1 c_1 = p_1[\xi_1 + \delta\gamma^{-1} f_0] = m_0$ holds, yielding (20a).

The cash-in-advance constraint binds at date t iff

$$v'(c_t) > \beta v'(c_{t+1})(p_t/p_{t+1}) \tag{A.1}$$

is satisfied. (19), (20), and (A.6) imply that (18) must hold in order for (10) to be an equality at all dates.

It remains to demonstrate that (17a) must hold. (17b), of course, follows from substituting (20a) in (16). For (17a) note that $p_t c_t = p_t \xi_t = m_1 = u_0 + (1-\delta)f_0$ holds $\forall\, t \geq 2$. Hence, for $t \geq 2, p_t = [u_0 + (1-\delta)f_0]/\xi_t$. Substituting this into (16) yields (17a). □

B. Proof of Proposition 2

Equations (8), (11), (23), and the fact that $n_t = \mu_t^f = 0\ \forall\, t > 1$ imply that

$$m_t = p_t \xi_t = (\gamma/\alpha)\xi_t;\ \ t \geq 2. \tag{A.2}$$

Then (10) at equality implies $c_{t+1} = \xi_t; t \geq 1$. Similarly, the date $t = 1$ version of (10) at equality gives $c_1 = (\alpha\gamma^{-1})m_0$. Hence (27) holds. Moreover, (25) must hold in order for the cash-in-advance constraint to bind at each date.

Equation (7) and $n_t = \mu_t^f = 0, t > 1$, imply that

$$\alpha\mu_t^u = \gamma(c_t - \xi_t);\ \ t > 1. \tag{A.3}$$

Then $\mu_t^u \geq 0,\ t \geq 2$, iff (24) holds.

For $t = 1$, equations (7), (27a), $n_1 = 0$, and $\mu_1^f = \delta f_0$ imply that

$$\alpha\mu_1^u = \alpha m_0 - \delta f_0 - \gamma\xi_1. \tag{A.4}$$

Hence $\mu_1^u \geq 0$ holds iff

$$(\alpha - \delta)f_0 + \alpha u_0 \geq \gamma\xi_1. \tag{A.5}$$

This completes the proof. □

C. Proof of Proposition 3

Equation (10) at equality, equations (8), (11) and (30), and $\mu_t^u \equiv 0$ imply that

$$m_t = \gamma\xi_t;\ \ t \geq 1. \tag{A.6}$$

Equation (10) at equality implies that

$$m_t = p_{t+1}c_{t+1} = \gamma c_{t+1};\ \ t \geq 0. \tag{A.7}$$

Hence (35) and $c_1 = \gamma^{-1}m_0$ hold.

These observations imply that (33) and (34) must hold in order for the cash-in-advance constraint to bind. The other requirement of an equilibrium is that $f_t \geq 0,\ \forall\, t \geq 1$. Since $m_1 = \gamma\xi_1, f_1 \geq 0$ requires $m_1 = \gamma\xi_1 \geq u_1$, or that $\gamma\xi_1 \geq u_0 + (1-\delta)f_0$. But this is just equation (31).

To derive (32) note that

$$u_t = u_{t-1} + (1-\delta)f_{t-1} = u_{t-1} + (1-\delta)(m_{t-1} - u_{t-1});\ \ t \geq 1. \tag{A.8}$$

(A.6) and (A.8) imply that

$$u_t = (1-\delta)\gamma\xi_{t-1} + \delta u_{t-1};\ \ t \geq 2. \tag{A.9}$$

The solution to (A.9) is

$$u_t = \gamma(1-\delta)\sum_{i=0}^{t-2}\delta^i\xi_{t-1-i} + \delta^{t-1}u_1;\ t \geq 2, \tag{A.10}$$

where $u_1 = u_0 + (1-\delta)f_0$. But then $m_t = \gamma\xi_t \geq u_t$ is equation (32). □

D. Proof of Proposition 4

Equation (10) at equality implies that $c_1 = \gamma^{-1}m_0$. Then, from (7), $n_1 - \mu_1^f = \gamma\xi_1 - m_0$. This is feasible iff $n_1 - \mu_1^f \geq -\delta f_0$, equivalently, iff

$$\gamma\xi_1 \geq (1-\delta)f_0 + u_0. \tag{A.11}$$

In odd periods (9b) and (36) imply that $\mu_t^u = 0$. Hence, from (8), (11), and (36), $m_t = p_t\xi_t = \gamma\xi_t$; t odd. (10) at equality and (36) imply that $m_t = p_{t+1}c_{t+1} = (\gamma/\alpha)c_{t+1}$; t odd. Hence (38*b*) holds.

To obtain (38c) note that equations (8), (11), (36), and $\mu_t^f = \delta f_{t-1}$; t even, imply that

$$m_t = (\gamma/\alpha)\xi_t + [\delta(1-\alpha)/\alpha]f_{t-1};\ t \text{ even}. \tag{A.12}$$

(A.12), (10) at equality, and (36) then yield (38c).

The evolution of f_t is obtained as follows. $f_1 = \delta f_0 + n_1 - \mu_1^f = \delta f_0 + \gamma\xi_1 - m_0 = \gamma\xi_1 - (1-\delta)f_0 - u_0$, as given in (37a). For t odd, $t > 1$, we have $f_t = \delta f_{t-1} + n_t - \mu_t^f$, and $f_{t-1} = 0$ if t is odd and $t > 1$. Therefore $\mu_t^f = 0$, t odd, $t > 1$, and $f_t = n_t$ holds $\forall\, t$ odd, $t > 1$.

These observations and $\mu_t^u = 0$, t odd, along with equation (7) yield that

$$n_t = \gamma(\xi_t - c_t);\ t \text{ odd}, t > 1. \tag{A.13}$$

Now use (38c) and $n_t = f_t$ (t odd, $t > 1$) in (A.13) to obtain

$$f_t = (\gamma/\alpha)(\alpha\xi_t - \xi_{t-1}) - [\delta(1-\alpha)/\alpha]f_{t-2} \equiv x_t - [\delta(1-\alpha)/\alpha]f_{t-2};\ t \text{ odd}, t > 1. \tag{A.14}$$

The solution to (A.14) is (37b). Feasibility requires that $f_1 \geq 0$ and $f_t \geq 0, t$ odd, $t > 1$ hold, with f_t as given by (A.14) for $t > 1$.

Equation (7), $n_t = 0$, and $\mu_t^f = \delta f_{t-1}$; t even, imply that

$$\alpha\mu_t^u = \gamma(c_t - \xi_t) - \delta f_{t-1} = \gamma(\alpha\xi_{t-1} - \xi_t) - \delta f_{t-1};\ t \text{ even} \tag{A.15}$$

holds, where the last line of (A.15) follows from (38b). Hence, (39) implies $\mu_t^u \geq 0,\ \forall\, t$.

It remains to show that $\mu_t^u \leq (1-\delta)f_{t-1} + u_{t-1}$; t even. To see this, use (A.15) to write the equivalent requirement that

$$\begin{aligned}(\gamma/\alpha)(\alpha\xi_{t-1} - \xi_t) &\leq [(\delta/\alpha) + (1-\delta)]f_{t-1} + u_{t-1}\\ &= [\delta(1-\alpha)/\alpha]f_{t-1} + f_{t-1} + u_{t-1}\\ &= [\delta(1-\alpha)/\alpha]f_{t-1} + m_{t-1}\\ &= [\delta(1-\alpha)/\alpha]f_{t-1} + \gamma\xi_{t-1};\ t \text{ even}.\end{aligned} \tag{A.16}$$

Since $f_{t-1} \geq 0\ \forall\, t$, (A.16) is obviously satisfied.

Finally, equations (36), (38), and (A.1) yield (40) as the condition that must be satisfied in order for the cash-in-advance constraint to bind. □

E. Construction of an economy with a 'Type 11' equilibrium

We now construct an economy that has a type 11 equilibrium in which the cash-in-advance constraint is binding at each date.

Since $\mu_t^{us} = \mu_t^{ug} = 0$; t even, (49), (58b) and (10) at equality imply that[32]

$$m_t = p_t\xi_t = \gamma_g\xi_t;\ t \text{ even}. \tag{A.17}$$

Then (10) at equality and (58c) imply that

$$c_{t+1} = \rho\xi_t;\ t \text{ even}. \tag{A.18}$$

Since there are no gold coins in odd periods ($\mu_t^{fg} = \delta_g f_{t-1}^g$, $\mu_t^{ug} = (1-\delta_g)f_{t-1}^g$; t odd), and since new silver is never coined, we have

$$m_t = u_1^s;\ t \text{ odd}. \tag{A.19}$$

Hence, from (10) at equality and (58b),

$$c_{t+1} = \gamma_g^{-1}u_1^s;\ t \text{ odd}. \tag{A.20}$$

Since $f_{t-1}^g = 0$ for all even values of t, we have $\mu_t^{fg} = 0$, t even. Thus, from (53) and our previous observations,

$$\gamma_g^{-1}n_t^g = \xi_t - c_t = \xi_t - \gamma_g^{-1}u_1^s;\ t \text{ even}.$$

Thus $n_t^g \geq 0, t$ even, requires that

$$\xi_t \geq \gamma_g^{-1}u_1^s;\ t \text{ even} \tag{A.21}$$

hold.

In odd periods (with $t > 1$) we have $n_t^s = n_t^g = \mu_t^{fs} = \mu_t^{us} = 0$, while

$$\mu_t^{fg} = \delta_g f_{t-1}^g = \delta_g n_{t-1}^g = \delta_g[\gamma_g\xi_{t-1} - u_1^s] \geq 0;\ t \text{ odd}, t > 1 \tag{A.22}$$

$$\mu_t^{ug} = (1-\delta_g)f_{t-1}^g = (1-\delta_g)n_{t-1}^g = (1-\delta_g)[\gamma_g\xi_{t-1} - u_1^s] \geq 0;\ t \text{ odd}, t > 1. \tag{A.23}$$

In addition, in odd periods (with $t > 1$), (53) implies that

$$-\gamma_g^{-1}(\mu_t^{fg} + \alpha_g\mu_t^{ug}) = \xi_t - c_t. \tag{A.24}$$

Therefore, (A.18), and (A.21)–(A.24) imply that

$$[\delta_g + \alpha_g(1-\delta_g)][\gamma_g^{-1}u_1^s - \xi_{t-1}] = \xi_t - \rho\xi_{t-1};\ t \text{ odd}, t > 1. \tag{A.25}$$

It remains to derive the equilibrium value for u_1^s. By definition,

$$u_1^s = u_0^s + (1-\delta_s)f_0^s - \mu_1^{us} = u_0^s - \mu_1^{us}, \tag{A.26}$$

since all full-weight silver is melted at $t = 1$. Furthermore $\mu_1^{fg} = \delta_g f_0^g$ and $\mu_1^{ug} = (1-\delta_g)f_0^g + u_0^g$ hold. (10) at equality and (58a) yield $c_1 = (\alpha_s\gamma_s^{-1})m_0$. These observations and (53) then imply that

[32] Note in particular that $f_t^s = 0\ \forall\ t \geq 1$, and hence $\mu_t^{fs} = 0, t > 1$.

$$(\alpha_s\gamma_s^{-1})(u_1^s - u_0^s) = \xi_1 - (\alpha_s\gamma_s^{-1})m_0 + \gamma_g^{-1}\delta_g f_0^g + \gamma_s^{-1}\delta_s f_0^s + (\alpha_g\gamma_g^{-1})[(1-\delta_g)f_0^g + u_0^g]. \tag{A.27}$$

If the value u_1^s given by (A.27) is positive and satisfies (A.21), then we may substitute this value into (A.25) and solve for ρ. If the solution satisfies (59), we have constructed an economy with the desired equilibrium.

It is easy to produce parameter values that yield the desired outcome. For example, let

$$\xi_1 = (\alpha_s\gamma_s^{-1})m_0 - \gamma_g^{-1}\delta_g f_0^g - \gamma^{-1}\delta_s f_0^s - (\alpha_g\gamma_g^{-1})[(1-\delta_g)f_0^g + u_0^g] > 0 \tag{A.28}$$

hold and in addition suppose that

$$\xi_t = \begin{cases} \xi_e; & t \text{ even} \\ \xi_0; & t \text{ odd}, t > 1 \end{cases} \tag{A.29}$$

Then (A.21) holds if $\xi_e \geq \gamma_g^{-1}u_0^s$. If we impose $\delta_g = 1$, then (A.25) reduces to

$$\rho = [\xi_0 + \xi_e - \gamma_g^{-1}u_1^s]/\xi_e. \tag{A.30}$$

Then (59) holds if

$$\alpha_g > [\xi_0 + \xi_e - \gamma_g^{-1}u_1^s]/\xi_e > \alpha_s\gamma_g/\gamma_s. \square \tag{A.31}$$

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